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A CLASS OF BRANCHING PROCESSES WITH TWO DEPENDENT TYPES*

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We propose a stochastic process model for a population of individuals of two types. Type-I individuals immigrate at the times of a Poisson process and have an arbitrary life time distribution. During their lives they generate type-II individuals, which for themselves may multiply and die. In contrast to classic branching processes they may have an influence on the life time of their type-I ancestor. Conversely at the death of the type-I ancestor all its type-II descendants die simultaneously. We derive the distributions of the relevant random variables and give conditions for the existence of limiting distributions. Finally some examples will be discussed.

Two-type branching process * depending types * limiting distributions

1. Introduction

In this paper we give the mathematical foundations of a two-type branching process, introduced by the authors in a former paper (Born and Dietz, 1989) in the context of a parasite-host population.

Our model is partly a specialization and partly a generalization of a two-type age-dependent branching process (cf. e.g. Mode (1971), Ch. 3). Type-I individuals enter the population by immigration only (and leave it by death). Type-II individuals are generated by type-I ancestors and multiply and die. We dispense with the assumption of independence of all individuals. The type-II descendants may have an influence on the lifetime of their type-I ancestor and will conversely die at its death. Only the independence of different type-I individuals and their descendants will be assumed.

We derive for every time t the distributions of the size of the type-I population, the size of the type-II population stemming from a randomly chosen living type-I

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individual and of the size of the type-II population stemming from all living type-I individuals. Conditions for the existence of limiting distributions of the above random variables are obtained. In the stationary case they exist if the life time expectancy of type-I individuals is finite. Finally we discuss different ways to choose the input distributions of our model appropriately, which gives insight to related papers.

A model like the above one has been successfully applied for the following epidemiological problem (cf. Born and Dietz (1989)). For some human infections, in particular those caused by certain species of macroparasites, the severity of the disease and hence the mortality of hosts (type-I individuals) do not depend only on the existence or absence of these parasites (type-II individuals) inside the host (as it is the case for virus induced diseases for instance) but on the number of parasites in the human body. This number depends on the immigration of parasites into the human host, which can be regarded as a form of generation of type-II individuals by type-I individuals, and the multiplication and death of parasites inside the host. Finally at the death of the host all its parasites die simultaneously. Of interest now are the distribution of the size of the parasite population in one host as a measure for the severity of the disease as well as the distribution of the parasite population size in all living hosts, which for instance determine the spread of the infection within the host population. These distributions may be the starting point for parameter identification by fitting and further statistical techniques from epidemiology.

The notation is as usual. $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the positive integers, the non negative integers, the real and the non negative real numbers, respectively. Let $\mathcal{M}^1(\Omega)=\mathcal{M}^1(\Omega,\mathfrak{A})$ be the space of probability measures on the measurable space $(\Omega,\mathfrak{A})$. For probability spaces $(\Omega_k,\mathfrak{A}_k,P_k)$, k from a non-empty set I, let $(\prod_{k\in I}\Omega_k,\bigotimes_{k\in I}\mathfrak{A}_k,\bigotimes_{k\in I}P_k)$ denote the corresponding product space. If $T:\Omega\to\Omega_k$, $S:\Omega\to\Omega_j$ are random variables $(k,j\in I)$, defined on a probability space $(\Omega,\mathfrak{A},P)$, then $P_T\in\mathcal{M}^1(\Omega_k)$ is the distribution of T and $T\times S:\Omega\to\Omega_k\times\Omega_j$ (or (T,S)) is the random variable $\omega\mapsto(T(\omega),S(\omega))$. The factorization of the conditional distribution $P_{T|S}$ of T with respect to S is denoted by $P_{T|S=s}$ $(s\in\Omega_j)$. Let $\Omega_k=\mathbb{R}$, T be integrable with respect to $P\in\mathcal{M}^1(\Omega)$ and $A\in\mathfrak{A}_j$. We then put $E(T|S\in A)=E(T|1_{[S\in A]}=1)$, where $1_{[S\in A]}(\omega)$ by definition is one for ω from the set $[S\in A]:=\{\omega\in\Omega: S(\omega)\in A\}$ and zero elsewhere. For $\Omega_k=\mathbb{Z}_+$ we can determine the distribution P_T of T by its probability generating function (pgf) $z\mapsto G_T(z)=E(z^T)$ $(z\in[0,1])$. The corresponding notation is used for conditional distributions on $\mathbb{Z}_+$.

2. The basic probability space

The probability space of our model is constructed in the usual way as follows (cf. Mode (1971), Ch. 3 and Jagers (1975), Ch. 6).

Let $\mathcal{I}^{\mathrm{I}}=\{\langle k\rangle: k\in\mathbb{N}\}$, where $\langle k\rangle$ is the kth immigrating type-I individual (instead of $\langle k\rangle$ we often write k). It establishes a type-II population $\mathcal{I}_k^{\mathrm{II}}=$

$\{\langle k, j_1, \ldots, j_n\rangle: n, j_1, \ldots, j_n \in \mathbb{N}\}$. Here $\langle k, j_1, \ldots, j_n\rangle$ denotes the j_nth child (of type II) of $\langle k, j_1, \ldots, j_{n-1}\rangle$ $(n \in \mathbb{N})$. An element of $\mathscr{I}_k^{\mathrm{II}}$ is denoted by i.

Now let $\mathfrak{X}$ be the path space of regular nonexplosive cadlag point processes. That is, $\mathfrak{X}$ is the set of all integer-valued Radon measures on $\mathbb{R}_+$, with the σ-algebra generated by the mappings $\mu \mapsto \mu(A)$ from $\mathfrak{X}$ into $\mathbb{R}_+$, for all measurable and bounded subsets A of $\mathbb{R}_+$ (cf. e.g. Jagers (1975), 5.4). Let

$\Omega_0 = \mathfrak{X}$ be the space of immigration times of type-I individuals;

$\Omega_i = \mathbb{R}_+ \times \mathfrak{X}$ be the space representing life length and age at births of descendants of type-II individual $i \in \bigcup_{k \in \mathscr{I}^{\mathrm{I}}} \mathscr{I}_k^{\mathrm{II}}$; and

$\Omega_k = \mathbb{R}_+ \times \mathfrak{X} \times \prod_{i \in \mathscr{I}_k^{\mathrm{II}}} \Omega_i$ be the space representing the life process of the kth type-I individual, where the first component gives its lifetime, the second one represents the age at births of type-II individuals with histories represented by the last component $(k \in \mathscr{I}^{\mathrm{I}})$.

Our basic measurable space now is $\Omega := \Omega_0 \times \prod_{k \in \mathscr{I}^{\mathrm{I}}} \Omega_k$ endowed with the corresponding product σ-algebra. On Ω we can define the random variables T_k, D_k as the time of immigration and the life length of type-I individual k, respectively, and $S_k(a)$ as the size of the type-II population stemming from type-I individual k at its age a $(k \in \mathbb{N},\ a \in \mathbb{R}_+)$. Our process will be represented by a probability measure $P = \bigotimes_{k \in \mathbb{Z}_+} P_k$ on Ω for $P_k \in \mathscr{M}^1(\Omega_k)$ $(k \in \mathbb{Z}_+)$ according to the following *basic assumptions*:

(a) $\{D_k, S_k(a): a \geq 0\}_{k \in \mathbb{N}}$ is a family of independent random vectors independent of $(T_i)_{i \in \mathbb{N}}$.

(b) P_{D_k} as well as $P_{S_k(a)|D_k > a}$, the latter represented by its pgf $G_{S(a)}^{D}$, are assumed to be known and independent of k as well as of the historical time t.

(c) The process $(T_j)_{j \in \mathbb{N}}$ is a Poisson process, not necessarily homogeneous, with intensity function $u \mapsto \nu(u)$.

(d) For technical reasons we assume that $T_1 > 0$, $S_k(0) = 0$ $(k \in \mathbb{N})$, type-I individuals immigrate with age 0 and for each type-I individual the process of the generated type-II population is cadlag.

The assumption of knowledge of P_D and $P_{S(a)|D>a}$ $(a \geq 0)$ may seem crucial. These distributions are in some sense intermediate distributions. What are usually taken as input distributions of a model like ours are the distributions $P_{D|S\equiv 0}$ of the lifetime of a type-I individual without any type-II descendants and the distributions $P_{S(a)}$ $(a \geq 0)$ of the size of the type-II population produced by a type-I individual irrespective if it is alive or not. It will be seen in Section 6 under additional assumptions how to derive P_D and $P_{S(a)|D>a}$ $(a \geq 0)$ from these distributions and how to construct the above probability space. In the following sections we obtain results without any further assumptions from the intermediate distributions P_D and $P_{S(a)|D>a}$ $(a \geq 0)$ concerning the distributions of the following three random variables, of most interest in our model $(t \geq 0,\ a \in [0, t])$:

$N(t; a)$, the number of individuals of type I, alive at time t with an age between 0 and a;

$S(t; a)$, the size of the type-II population of a type-I individual, which is chosen at random from all type-I individuals alive at time t with an age between 0 and a;

$X(t;a)$, the size of the total type-II population at time t, stemming from living type-I individuals with an age less than a.

We point out that $N(t;a)$ as well as $S(t;a)$ and $X(t;a)$ concerns individuals with an age between 0 and a, that is, born between $t-a$ and t, while in $S(a)$ the letter a refers to a fixed age. Furthermore $S(t;a)$ indeed represents a conditionally distributed random variable, which will be made precise in Section 3.

In what follows, t will always be from $\mathbb{R}_+$, denoting historical time, while $a\in[0,t]$ denotes age, and z will be from $[0,1]$.

3. The size of the type-I population

The following proposition can be obtained via the transformation $u\mapsto\tilde{u}=\int_{t-a}^{u}\nu(v)\,\mathrm{d}v$ for $u\geq t-a$ from the well known result on the conditional distributions of entering times in a homogeneous Poisson process (cf. Karlin (1966), 7.3.2).

3.1. Proposition. *Let $M(t;a):=\sum_{k\in\mathbb{N}}1_{[t-a\leq T_k<t]}$ be the number of immigrants (of type* I*) between $t-a$ and t and $M(t):=M(t;t)$. Then for all $n\in\mathbb{N}$ we have*

$$P_{T_{M(t-a)+1},\dots,T_{M(t)}|M(t;a)=n}=P_{U_{(1)},\dots,U_{(n)}},$$

where $(U_1,\dots,U_n)$ is an i.i.d. family of random variables with

$$P(U_i\leq u)=1_{[t-a,t]}(u)\int_{t-a}^{u}\nu(v)\,\mathrm{d}v\left(\int_{t-a}^{t}\nu(u)\,\mathrm{d}u\right)^{-1}$$

and $(U_{(1)},\dots,U_{(n)})$ is the corresponding ordered sample.

3.2. Remark. In the situation of Proposition 3.1 let $h:\mathbb{R}^n\to\mathbb{R}$ be symmetric; i.e. $h(u_{\pi(1)},\dots,u_{\pi(n)})=h(u_1,\dots,u_n)$ for all $(u_1,\dots,u_n)\in\mathbb{R}^n$ and all π from S_n, the symmetric group of order n. Then

$$\int_{\mathbb{R}^n}h\,\mathrm{d}P_{U_{(1)},\dots,U_{(n)}}=\int_{\mathbb{R}^n}h\,\mathrm{d}P_{U_1,\dots,U_n}.$$

Proof. For $\pi\in S_n$ let $\Omega_\pi:=\{\omega\in\Omega:\ U_{\pi(k)}=U_{(k)}\ (1\leq k\leq n)\}$. Then $(\Omega_\pi)_{\pi\in S_n}$ defines a partition up to sets of P-measure zero of Ω (observe the distribution of U_i). □

3.3. Theorem. *Let $N(t;a):=\sum_{k\in\mathbb{N}}1_{[t-a\leq T_k<t<T_k+D_k]}$ be the number of type-*I *individuals living at time t with an age not exceeding a, $N(t):=N(t;t)$. Then $N(t;a)$ is Poisson distributed with expectation*

$$E(N(t;a))=\int_{t-a}^{t}P(D>t-u)\nu(u)\,\mathrm{d}u=\int_{0}^{a}P(D>u)\nu(t-u)\,\mathrm{d}u.$$

Proof. By the basic assumptions, Proposition 3.1 and Remark 3.2 we have

$$
\begin{aligned}
E(z^{N(t;a)}) &= \int_{\mathbb{Z}_+} \int_{\mathbb{R}_+^m} \int_{\mathbb{R}_+^m} \prod_{k=1}^m z^{1_{[t<t_k+d_k]}} \, \mathrm{d}P_{\otimes_{k=1}^m D_k}((d_k)_{k\le m}) \\
&\qquad \cdot \mathrm{d}P_{\otimes_{k=1}^m T_{M(t-a)+k}|M(t;\,a)=m}((t_k)_{k\le m}) \, \mathrm{d}P_{M(t;\,a)}(m) \\
&= \sum_{m\in\mathbb{Z}_+} \int_{\mathbb{R}_+^m} \prod_{k=1}^m E(z^{1_{[t<u_k+D_k]}}) \\
&\qquad \cdot \mathrm{d}P_{U_{(1)},\dots,U_{(m)}}(u_1,\dots,u_m) P(M(t;a)=m) \\
&= \sum_{m\in\mathbb{Z}_+} (E(z^{1_{[t<U+D]}}))^m P(M(t;a)=m) \\
&= G_{M(t;\,a)}(E(z^{1_{[t<U+D]}})),
\end{aligned}
\tag{3.4}
$$

where the U_i are distributed according to Proposition 3.1. The assertion now follows from $(T_k)_{k\in\mathbb{N}}$ being a Poisson process, hence $G_{M(t;a)}(z) = \exp[-(1-z)\int_{t-a}^t \nu(u)\,\mathrm{d}u]$ and

$$
E(z^{1_{[t<U+D]}}) = (z-1)\int_{t-a}^t P(D>t-u)\frac{\nu(u)}{\int_{t-a}^t \nu(v)\,\mathrm{d}v}\,\mathrm{d}u + 1. \qquad \square \tag{3.5}
$$

Obviously $(N(t))_{t\ge 0}$ is in general not Markovian but a queueing process (cf. Takacs (1962), Ch. 3).

By replacing in the proof of Theorem 3.3 the set $[t<T_k+D_k]$ by $[t<T_k+D_k]\cap[S_k(t-T_k)=r]$ we obtain

3.6. Theorem. *For $r\in\mathbb{Z}_+$ the number*

$$
N(t;a,r) := \sum_{n\in\mathbb{N}} 1_{[t-a\le T_k<t<T_k+D_k]} 1_{[S_k(t-T_k)=r]}
$$

*of type-*I *individuals living at time t with an age not exceeding a and a descending type-*II *population of size r is Poisson distributed with expectation*

$$
E(N(t;a,r)) = \int_0^a P(S(u)=r|D>u)P(D>u)\nu(t-u)\,\mathrm{d}u. \tag{3.7}
$$

4. The type-II population of a type-I ancestor

Of most importance in what follows is the distribution of the size of the type-II population stemming from a type-I individual, which is chosen at random at time t from all living type-I individuals with age less than a. This can be expressed as follows.

Let $n(t;a) := \{k\in\mathbb{N}: t-a\le T_k<t<T_k+D_k\}$ be the set of indices of type-I individuals alive at time t and with an age not exceeding a. It depends measurably on ω and is finite P-a.s. according to the assumptions. For every $\omega\in\Omega$ let $Q=Q(\omega)$

be the discrete uniform distribution on the random set $n(t;a)$ if $n(t;a)\neq\emptyset$, and the Dirac measure in the point infinity (∞) otherwise. In this sense it is a random measure on $\mathbb{Z}_+$. The identity $k(t;a)$ on $(\mathbb{Z}_+, Q(\cdot))$ then represents the random choice of a type-I individual from all living type-I individuals with an age not exceeding a under the condition $[n(t;a)\neq\emptyset]$. It is independent of all other events which have happened up to time t. For technical reasons put $T_\infty=\infty$, $D_\infty=0$. Now the pgf $G_{S(t;a)}$ of the size of the type-II population stemming from a type-I individual, which is chosen at random at time t from all living type-I individuals with age less than a is given by

$$z\mapsto G_{S(t;a)}(z) := E(z^{S_{k(t;a)}(t-T_{k(t;a)})}\mid N(t;a)>0).$$

It can be derived from our intermediate distributions P_D and $P_{S(a)\mid D>a}$ $(a\geq 0)$ as follows.

4.1. Theorem. *We have*

$$G_{S(t;z)}(z) := \int_0^a G^D_{S(u)}(z)\,\frac{P(D>u)\nu(t-u)}{\int_0^a P(D>v)\nu(t-v)\,\mathrm{d}v}\,\mathrm{d}u.$$

Proof. From our basic assumptions obviously

$$\begin{aligned} G_{S(t;a)}(z) &= \int_{t-a}^{t} E(z^{S_{k(t;a)}(t-T_{k(t;a)})}\mid T_{k(t;a)}=u,\ N(t;a)>0)\\ &\qquad\cdot P_{T_{k(t;a)}\mid N(t;\,a)>0}(\mathrm{d}u)\\ &= \int_{t-a}^{t} G^D_{S(t-u)}(z)\,P_{T_{k(t;a)}\mid N(t;\,a)>0}(\mathrm{d}u). \end{aligned} \tag{4.2}$$

To obtain an explicit expression for $P_{T_{k(t;a)}\mid N(t;\,a)>0}$, take a measurable set $A\subseteq [t-a, t]$. With the notations from Proposition 3.1 and the proof of Theorem 3.3 we obtain

$$\begin{aligned} &P(T_{k(t;a)}\in A,\ N(t;a)>0)\\ &= \sum_{m\in\mathbb{Z}_+}\sum_{n=1}^{m}\sum_{|B|=n}\sum_{j\in B} P(k(t;a)=j\mid n(t;a)=B,\ T_j\in A)\\ &\qquad\cdot P(n(t;a)=B,\ T_j\in A)\\ &= \sum_{m\in\mathbb{N}}\sum_{n=1}^{m}\frac{1}{n}\sum_{|B|=n}\sum_{j\in B}\int_{\mathbb{R}_+^m}\int_{\mathbb{R}_+^m} 1_A(t_j)\prod_{i\in B} 1_{]t,\infty[}(t_i+d_i)\\ &\quad\cdot\prod_{i\notin B} 1_{[0,t]}(t_i+d_i)\,\mathrm{d}P_{\otimes_{k=1}^m D_k}((d_k)_{k\leq m})\\ &\quad\cdot\mathrm{d}P_{\otimes_{k=1}^m T_{M(t-a)+k}\mid M(t;a)=m}((t_k)_{k\leq m})P(M(t;a)=(m))\\ &= \sum_{m\in\mathbb{N}}\sum_{n=1}^{m}\frac{1}{n}\int_{\mathbb{R}_+^m}\left\{\sum_{|B|=n}\sum_{j\in B} 1_A(u_j)\prod_{i\in B}P(D>t-u_i)\prod_{i\notin B}P(D\leq t-u_i)\right\}\\ &\qquad\cdot\mathrm{d}P_{U_{(1)},\dots,U_{(m)}}(u_1,\dots,u_m)P(M(t;a)=m), \end{aligned} \tag{4.3}$$

where the third sum always extends over all subsets B of $\{1, \ldots, m\}$ with n elements, the second product runs over the set $\{1, \ldots, m\} \setminus B$ and U_i is distributed according to Proposition 3.1.

Let $f: \mathbb{R}^m \to \mathbb{R}$ denote the integrand. Then we are left to show that f is symmetric, because then (cf. Remark 3.2)

$$
\begin{aligned}
&P(T_{k(t;a)} \in A,\ N(t;a) > 0) \\
&= \sum_{m \in \mathbb{N}} \sum_{n=1}^{m} \frac{1}{n} \left(\int_{\mathbb{R}_+^m} f \, \mathrm{d}P_{U_1,\ldots,U_m} \right) P(M(t;a) = m) \\
&= \int_A P(D > t-u) \frac{\nu(u)}{\int_{t-a}^{t} \nu(v)\,\mathrm{d}v} \,\mathrm{d}u \\
&\cdot \left\{ \sum_{m \in \mathbb{N}} \sum_{n=1}^{m} \left(\int_{[t-a,t]} P(D > t-u) \frac{\nu(u)}{\int_{t-a}^{t} \nu(v)\,\mathrm{d}v} \,\mathrm{d}u \right)^{n-1} \right. \\
&\cdot \left. \left(\int_{[t-a,t]} P(D \leq t-u) \frac{\nu(u)}{\int_{t-a}^{t} \nu(v)\,\mathrm{d}v} \,\mathrm{d}u \right)^{m-n} P(M(t;a) = m) \right\}, \qquad (4.4)
\end{aligned}
$$

which implies

$$
P_{T_{k(t;a)} | N(t;a) > 0}(A) = \frac{\int_A P(D > t-u)\nu(u)\,\mathrm{d}u}{\int_{t-a}^{t} P(D > t-u)\nu(u)\,\mathrm{d}u}.
$$

From this the assertion follows.

To show that f is symmetric let S_m be again the symmetric group of order m and

$$
S_m^f := \{\pi \in S_m : f(\pi(u)) = f(u) \text{ for all } u \in \mathbb{R}^m\},
$$

where $\pi(u) := (u_{\pi(1)}, \ldots, u_{\pi(m)})$ for $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ and $\pi \in S_m$. Obviously S_m^f is a subgroup of S_m. We will show that S_m^f contains all transpositions, hence $S_m^f = S_m$ and f is symmetric. To this end write $f(u) = \sum_{|B|=n} g_B(u)$ with

$$
g_B(u) = \sum_{j \in B} 1_A(u_j) \prod_{i \in B} P(D > t-u_i) \prod_{i \notin B} P(D \leq t-u_i),
$$

notations as above. Let $\pi \in S_m$ with $\pi(i) = j$, $\pi(j) = i$, $\pi(k) = k$ for some $i, j \in \{1, \ldots, m\}$ and all $k \notin \{i, j\}$. If both $i, j \in B$ or $i, j \notin B$ then g_B obviously is symmetric with respect to π. For $\binom{n-1}{m-2}$ subsets B_j^i of $\{1, \ldots, m\}$ we have $j \in B_j^i$, $i \notin B_j^i$, then $g_{B_j^i}(\pi(u)) = g_{B_i^j}(u)$ for all $u \in \mathbb{R}^m$ and as many subsets B_i^j of $\{1, \ldots, m\}$ with $i \in B_i^j$ and $j \notin B_i^j$. But as well $g_{B_i^j}(\pi(u)) = g_{B_j^i}(u)$ for all $u \in \mathbb{R}^m$ and $\binom{n-1}{m-2}$ subsets B_i^j of $\{1, \ldots, m\}$ with $i \in B_i^j$ and $j \notin B_i^j$. Hence $f(\pi(u)) = f(u)$ for all $u \in \mathbb{R}^m$. □

Now the random variable $S(t;a)$ introduced in Section 2 has to be understood as a random variable, the distribution of which by definition is generated by $G_{S(t;a)}$. It is the mixture of the distributions $P_{S(a)|D>a}$ $(a \in \mathbb{R}_+)$ with respect to the distribution $P_{T_{k(t;a)}|N(t;a)>0}$ of the entering time of a type-I individual, chosen by random from all type-I individuals alive at time t with an age not exceeding a. We point out that $S(t;a)$ in general cannot be defined on the original probability space Ω.

5. Total number of type-II individuals

The third object of interest is the number

$$X(t;a) := \sum_{k=M(t-a)+1}^{M(t)} S_k(t-T_k)\cdot 1_{[t<T_k+D_k]}$$

of all living type-II individuals stemming from type-I individuals, which are alive at time t with an age not exceeding a. Again $X(t)=X(t;t)$. The methods applied in the preceding sections work as well to obtain the pgf of the corresponding distribution.

5.1. Theorem. *We have* $G_{X(t;a)}(z)=G_{N(t;a)}(G_{S(t;a)}(z))$.

Proof. The arguments from the preceding proofs and the result of Theorem 4.1 imply the equality

$$\begin{aligned}
G_{X(t;a)}(z) &= E(z^{X(t;a)})\\
&= \sum_{k,m,n,B} z^k P\Big([M(t;a)=m]\cap\bigcap_{i\in B}[T_i+D_i>t]\\
&\qquad \cap\bigcap_{i\notin B}[T_i+D_i\le t]\cap\Big[\sum_{i\in B}S_i(t-T_i)=k\Big]\Big)\\
&= \sum_{k,m,n,B} z^k\int 1_{\{k\}}\Big(\sum_{i\in B}S_i(t-T_i)\Big)\prod_{i\in B}1_{]t,\infty[}(T_i+D_i)\\
&\qquad \cdot\prod_{i\notin B}1_{[0,t]}(T_i+D_i)1_{\{m\}}(M(t;a))\,\mathrm{d}P\\
&= \sum_{m,n,B}\int_{\mathbb{R}_+^m}\int_{\mathbb{R}_+^m}\int_{\mathbb{Z}_+^m} z^{(\sum_{i\in B}s_i)}\prod_{i\in B}1_{]t,\infty[}(t_i+d_i)\\
&\quad\cdot\prod_{i\notin B}1_{[0,t]}(t_i+d_i)\,\mathrm{d}\Big(\bigotimes_{i\in B}P_{S_i(t-t_i)|D_i=d_i}\Big)((s_i)_{i\in B})\\
&\quad\cdot\mathrm{d}\Big(\bigotimes_{i=1}^m P_{D_i}\Big)((d_i)_{i\le m})\,\mathrm{d}\Big(\bigotimes_{i=1}^m P_{U_i}\Big)((t_i)_{i\le m})P(M(t;a)=m)\\
&= \sum_{m,n,B}\prod_{i\in B}\Big(\int_{t-a}^t E(z^{S_i(t-u)}1_{]t,\infty[}(u+D))\frac{\nu(u)}{\int_{t-a}^t\nu(v)\,\mathrm{d}v}\mathrm{d}u\Big)\\
&\quad\cdot\prod_{i\notin B}P(0\le U_i+D_i\le t)P(M(t;a)=m)\\
&= \sum_{m,n,B}\Big(\int_{t-a}^t\sum_{k\in\mathbb{Z}_+}z^kP(S_i(t-u)=k,D>t-u)\\
&\qquad\cdot\frac{\nu(u)}{\int_{t-a}^t\nu(v)\,\mathrm{d}v}\frac{P(D>t-u)}{P(D>t-u)}\mathrm{d}u\Big)^n
\end{aligned}$$

$$\cdot \prod_{i \notin B} P(0 \leq U_i + D_i \leq t) P(M(t; a) = m)$$

$$= \sum_{n \in \mathbb{Z}_+} \sum_{m=n}^{\infty} \sum_{|B|=n} \left(\int_{t-a}^{t} G^D_{S(t-u)}(z) \frac{P(D > t-u)\nu(u)}{\int_{t-a}^{t} P(D > t-v)\nu(v)\,\mathrm{d}v} \mathrm{d}u \right)^n$$

$$\cdot \left(\int_{t-a}^{t} P(D > t-u) \frac{\nu(u)}{\int_{t-a}^{t} \nu(v)\,\mathrm{d}v} \mathrm{d}u \right)^n$$

$$\cdot \prod_{i \notin B} P(0 \leq U_i + D_i \leq t) P(M(t; a) = m)$$

$$= \sum_{n \in \mathbb{Z}_+} (G_{S(t;a)}(z))^n P(N(t; a) = n)$$

$$= G_{N(t;a)}(G_{S(t;a)}(z)), \tag{5.2}$$

where $\sum_{k,m,n,B}$ extends over all $k \in \mathbb{Z}_+$ (if listed), $m \in \mathbb{Z}_+$, $n \in \{0, 1, \ldots, m\}$ and all subsets B of $\{0, 1, \ldots, m\}$ with n elements (cf. also the notations of (4.3)). □

By this result the distribution of $X(t; a)$ is obtained by compounding the distributions $P_{S(t;a)}$ of the different individual type-II populations sizes and the Poisson distribution of the number of these populations, which equals the size of the living type-I population.

A completely different and maybe somewhat more simple derivation of this theorem for constant ν and $a = t$ relies on the following integral equation, which shows the close relation to multitype branching processes.

5.3. Lemma. *Let $n := M(t-a) + 1$. Then*

$$G_{X(t;a)}(z) = P(T_n \geq t) + \int_{t-a}^{t} \{P(D \leq t-u) + P(D > t-u) G^D_{S(t-u)}(z)\} \cdot G_{X(t;t-u)}(z) P_{T_n}(\mathrm{d}u). \tag{5.4}$$

Proof. Put $Z_1 := T_n$ and $Z_j := T_{n+j} - T_{n+j-1}$ for $j \geq 2$. Then $(Z_j)_{j \in \mathbb{N}}$ is a family of independent random variables, independent of $(S_k(a))_{a \geq 0}$ for all $k \in \mathbb{N}$ and

$$G_{X(t;a)}(z) = \sum_{r \in \mathbb{Z}_+} z^r \{P(X(t; a) = r, T_n \geq t) + P(X(t; a) = r, T_n < t)\}$$

$$= P(T_n \geq t) + \int_{t-a}^{t} E\left(z^{\sum_{k \geq n} S_k(t-T_k) \cdot 1_{[T_k < t < T_k + D_k]}} \middle| T_n = u\right) P_{T_n}(\mathrm{d}u)$$

$$= P(T_n \geq t) + \int_{t-a}^{t} E\left(z^{S_n(t-u) \cdot 1_{[t-u < D_n]}}\right) \cdot E\left(z^{\sum_{k > n} S_k(t-u-Z_2-\cdots-Z_k) \cdot 1_{[Z_2+\cdots+Z_k < t-u < Z_2+\cdots+Z_k+D_k]}} \middle| T_n = u\right) \cdot P_{T_n}(\mathrm{d}u). \tag{5.5}$$

By our basic assumptions the second expectation in the integral is $G_{X(t;t-u)}(z)$. The first equals $P(D \leq t-u) + P(D > t-u) G^D_{S(t-u)}(z)$. □

5.6. Corollary. *If $(T_k)_{k\in\mathbb{N}}$ is a homogeneous Poisson process (i.e. ν is constant), then*

$$G_{X(t)}(z)=\exp\left[-\int_0^t \nu P(D>u)(1-G^D_{S(u)}(z))\,\mathrm{d}u\right]$$

$$=G_{N(t)}(G_{S(t;t)}(z)). \tag{5.7}$$

Proof. Under the assumptions $P_{X(t;t-u)}$ depends only on $t-u$, hence Lemma 5.3 for $a=t$ reads

$$G_{X(t)}(z)=\mathrm{e}^{-\nu t}+\mathrm{e}^{-\nu t}\int_0^t \{P(D\leq u)+P(D>u)G^D_{S(u)}(z)\}G_{X(u)}(z)\,\mathrm{d}u.$$

It follows that

$$\frac{\partial}{\partial t}G_{X(t)}(z)=G_{X(t)}(z)\nu P(D>t)(G^D_{S(t)}(z)-1),\quad G_{X(0)}\equiv 1,$$

which has a unique solution

$$G_{X(t)}(z)=\exp\left[-\int_0^t \nu P(D>u)(1-G^D_{S(u)}(z))\,\mathrm{d}u\right].$$

Theorem 3.3 and Theorem 4.1 (both for constant ν) imply the second equality of the assertion. □

6. Existence of limiting distributions

The limiting distributions of the above random variables for time t tending to infinity are of particular interest in many applications. They are the stationary distributions in the sense that they do not change in time.

For the existence of these limiting distributions there is a difference between the age-related random variables $N(t;\,a)$, $S(t;\,a)$ and $X(t;\,a)$ and the non-age-related random variables $N(t)$, $S(t)$ and $X(t)$. The distribution of the age-related random variables depend on the values of the immigration intensity ν of type-I individuals in the interval $[t-a,\,t]$ only. Hence a convergence assumption on ν is sufficient to ensure the existence of limiting distributions. For the non-age-related random variables this is not sufficient.

6.1. Theorem. *Let the immigration intensity ν of type-*I *individuals be convergent, $\nu_\infty:=\lim_{t\to\infty}\nu(t)$. Then $P_{N(t;a)}$, $P_{S_{k(t;a)}(t-T_{k(t;a)})|N(t;a)>0}$ and $P_{X(t;a)}$ are weakly convergent for t tending to infinity, the limiting distributions are probability distributions on $\mathbb{Z}_+$. If in addition ν is bounded and $E(D)$ is finite, then the assertion holds as well for $P_{N(t)}$, $P_{S_{k(t;t)}(t-T_{k(t;t)})|N(t;t)>0}$* and *$P_{X(t)}$.*

Proof. As in the continuity theorem for Fourier transforms we only have to show that the corresponding pgf's are pointwise convergent for t tending to infinity to a limit, continuous in $z=1$.

Let us first assume only that ν is convergent. Then $\nu_t : u \mapsto \nu(t-u)$ converges to constant ν_∞ uniformly on $[0, a]$. Hence

$$E(N(\infty;\, a)) := \lim_{t\to\infty} \int_0^a P(D>u)\nu(t-u)\,du$$

exists and

$$G_{N(t;\,a)}(z) = \exp\left[-(1-z)\int_0^a P(D>u)\nu(t-u)\,du\right]$$

is convergent pointwise and indeed uniformly on $[0, 1]$.

By the same argument

$$\begin{aligned} &G_{S(t;a)}(z) \\ &\quad= \left(\int_0^a G^D_{S(u)}(z)P(D>u)\nu(t-u)\,du\right) \Big/ \left(\int_0^a P(D>u)\nu(t-u)\,du\right) \end{aligned}$$

converges pointwise to

$$G_{S(\infty;a)}(z) := \left(\lim_{t\to\infty}\int_0^a G^D_{S(u)}(z)P(D>u)\nu(t-u)\,du\right)E(N(\infty;\,a))^{-1}.$$

But $|G_{S(u)}(z)| \le 1$ for all $z\in[0, 1]$; hence a short calculation shows that $|G_{S(\infty,a)}(z) - G_{S(t;\,a)}(z)|$ is small for t sufficiently large independent of z, so the convergence is uniformly in z and the assertion follows.

The convergence of $(G_{N(t;a)})_{t\ge0}$ and $(G_{S(t;a)})_{t\ge0}$ uniformly on $[0, 1]$ finally implies that $G_{X(t;a)} = G_{N(t;a)} \circ G_{S(t;a)}$ is convergent for t tending to infinity uniformly on $[0, 1]$.

Let now in addition ν be bounded and $E(D)$ be finite. Then $E(N(\infty)) := \lim_{t\to\infty}\int_0^t P(D>u)\nu(t-u)\,du$ exists: If t_D and t_ν are chosen such that $\int_{t_D}^\infty P(D>u)\,du < \varepsilon$ for $t>t_D$ and $|\nu(t)-\nu_\infty| < \varepsilon$ for $t>t_\nu$, then

$$\begin{aligned} &\left|\int_0^{t+h} P(D>u)\nu(t+h-u)\,du - \int_0^t P(D>u)\nu(t-u)\,du\right| \\ &\quad\le \left|\int_t^{t+h} P(D>u)\nu(t+h-u)\,du\right| \\ &\qquad + \left|\int_0^{t-t_\nu} P(D>u)\{\nu(t+h-u)-\nu(t-u)\}\,du\right| \\ &\qquad + \left|\int_{t-t_\nu}^{t} P(D>u)\{\nu(t+h-u)-\nu(t-u)\}\,du\right| \\ &\quad\le \varepsilon\left(\sup_{t\in\mathbb{R}_+}\nu(t) + E(D) + 2\sup_{t\in\mathbb{R}_+}\nu(t)\right) \end{aligned} \tag{6.2}$$

holds for all $t > t_\nu + t_D$ and $h>0$.

The remaining assertions follow by the same arguments as above. □

Obviously a constant ν satisfies all the above assumptions on ν.

The basic properties of the distributions of the above random variables for finite t remain valid if t tends to infinity. For both $N(t;a)$ and $N(t)$ the limiting distributions are Poisson, the first one with expectation $\int_0^a P(D>u)\nu_\infty\,du$, the expectation of the second one is bounded by $E(D)\cdot\sup_{t\geq 0}\nu(t)$. The limiting distributions of $S(t;a)$ and $S(t)$ again are mixtures of the laws of branching processes with respect to the entering time distribution of living type-I individuals. The compounding formula for $P_{X(t;a)}$ finally remains valid for the limiting distribution. Hence it is also a compound distribution as is described in Section 5. If $E(S(t;a))$ and $V(S(t;a))$ denote expectation respectively variance (possibly infinite) of the distribution determined by $G_{S(t;a)}$, then by differentiating $G_{X(t;a)}$ with respect to z we obtain

$$E(X(t;a))=E(N(t;a))E(S(t;a))$$

and

$$V(X(t;a))=E(N(t;a))\{V(S(t;a))+E(S(t;a))^2\}$$

in the usual meaning. This implies that the dispersion index $V(X(t;a))/E(X(t;a))$ of $P_{X(t;a)}$ is not smaller than one, which is the dispersion index of a Poisson distribution.

Under the above assumptions the limiting distributions do not have any mass at infinity. Nevertheless they do not need to have finite moments. More details about them are available only if the input distributions of our model are precisely determined. General concepts for this are discussed in the next section, while complete calculations can be found in Born and Dietz (1989), Dietz (1982), Hadeler and Dietz (1984), Pakes (1986) and Tessera (1984).

7. Examples and applications

In this section we show how to construct particular classes of two-type branching processes, for which the intermediate distributions P_D and $P_{S(a)|D>a}$ $(a\geq 0)$ can be derived explicitly.

First of all let the type-II individuals and their ancestors be independent. Then $P_{S(a)|D>a}=P_{S(a)}$ for all $a\geq 0$, that is, the dynamics of the size of the type-II population stemming from a type-I ancestor does not depend on its life situation. This does not change the definition of X: for this random variable the life of a type-II individual still ceases at the end of the life of the type-I ancestor. The measures P_k on Ω_k, $k\in\mathbb{N}$, appearing in the definition of the basic probability space in Section 2, are of the form

$$P_k=P_k^{\mathrm{I}}\otimes P_k^{\mathrm{II}}$$

for measures $P_k^{\mathrm{I}}\in\mathcal{M}^1(\mathbb{R}_+\times\mathfrak{X})$ (law of the ancestor),

and $P_k^{\mathrm{II}}\in\mathcal{M}^1(\prod_{i\in\mathcal{I}_k^{\mathrm{II}}}\Omega_i)$ (law of its type-II descendants)

$(k\in\mathbb{N})$.

There are many ways to choose these measures appropriately. P_k^{I} can be any law determining the life length of type-I individual k and the times of birth respectively acquisition of type-II individuals (cf. Jagers (1975), Ch. 6 and Mode (1971), Ch.3). P_k^{II} may be of the form

$$P_k^{\mathrm{II}} = \bigotimes_{i \in \mathcal{I}_k^{\mathrm{II}}} P_i^{\mathrm{II}} \quad \text{for measures } P_i^{\mathrm{II}} \in \mathcal{M}^1(\Omega_i),$$

defined according to (Mode (1971), 3.2). This leads to a particular case of an ordinary generalized age depending two-type branching process.

Under the same assumptions we may as well define

$$P_k = P_D \otimes P_k^{\mathrm{II}} \quad \text{for } P_D \in \mathcal{M}^1(\mathbb{R}_+),\ P_k^{\mathrm{II}} \in \mathcal{M}^1\Big(\mathcal{X} \times \prod_{i \in \mathcal{I}_k^{\mathrm{II}}} \Omega_i\Big),$$

the latter being the law of a Markovian immigration-birth-death process (cf. Bailey (1964), 8.7, 9.4, and Karlin (1966)). This approach is applied for instance in a paper of Tessera (1984) (cf. also Pakes (1986)) within a model for a human population with immigration and multiplication of individuals, and emigration of tribes. For all choices of the distributions P_k^{I} and P_k^{II} the results of the preceding sections are valid with $P_{S(a)|D>a} = P_{S(a)}$ and $P_{D|S\equiv 0} = P_D$.

If the type-II individuals and their ancestors are not necessarily independent, one can construct the measures P_k on Ω_k as the laws of a regular Markov process $(1_{[D_k>a]}, S_k(a))_{a\geq 0}$ with state space $\{0, 1\} \times \mathbb{Z}_+$ and a suitable intensity function q defined by

$$q_{i,m;j,n}(a) = \lim_{h \downarrow 0} \frac{1}{h} P(1_{[D_k>a+h]} = i,\ S_k(a+h) = m \mid 1_{[D_k>a]} = j,\ S_k(a) = n).$$

This was done in Born and Dietz (1989) in the above mentioned context of a parasite-host population. They put

$$q_{i,m;j,n}(a) := \begin{cases} \mu + n \cdot \alpha & \text{for } i=1, j=0,\ n, m \in \mathbb{Z}_+,\ m = n, \\ n \cdot \sigma & \text{for } i=1, j=1,\ n, m \in \mathbb{N},\ m = n-1, \\ \varphi + n \cdot \rho & \text{for } i=1, j=1,\ n, m \in \mathbb{Z}_+,\ m = n+1, \\ -\mu - \varphi - n(\alpha + \sigma + \rho) & \text{for } i=1, j=1,\ n, m \in \mathbb{Z}_+,\ m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Here φ, ρ and σ are the intensities of immigration, birth and death respectively of a type-II individual, μ is the dying intensity of type-I individuals and α describes the (linear) dependence of the life length of a type-I individual from the size of the generated type-II population (cf. also Dietz (1982), p. 100 and Hadeler and Dietz (1984), Sect. 4) The distribution P_D then is given by

$$P(D > a) = \exp\left[-\int_0^a \mu + \alpha E(S(u) \mid D > u)\, \mathrm{d}u \right].$$

The explicit form of $P_{S(a)|D>a}$ ($a \geq 0$) is already complicated and can be found in Born and Dietz (1989, Sect. 5).

Finally we point out that another way of looking at the dynamics of our total type-II population stemming from living type-I ancestors is that of catastrophes, occurring at the times of death of the ancestors. The decrease of type-II population then is just $S_k(D_k)$. For a deeper discussion of this aspect we refer to Brockwell et al. (1982).

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